6.1 Maxima and Minima

How can the owner of a snowboard rental business use mathematics to maximize sales or minimize cost? What dimensions of a rectangular field provide the greatest area? Questions like these are answered by finding the maximum or minimum point of a quadratic relation, which occurs at the vertex.

If a relation is of the form $y = a(x - h)^2 + k$, then the vertex is (h, k). However, if a relation is of the form $y = ax^2 + bx + c$, the coordinates of the vertex are not so obvious. In this section, you will learn how to express $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$.

Investigate

Tools
- algebra tiles
- graphing calculator

How can you model the process of creating a perfect square?

1. Consider the quadratic expression $x^2 + 6x + 9$.

a) Show that the expression is a perfect square using algebra tiles.

b) Factor the expression as a perfect square.

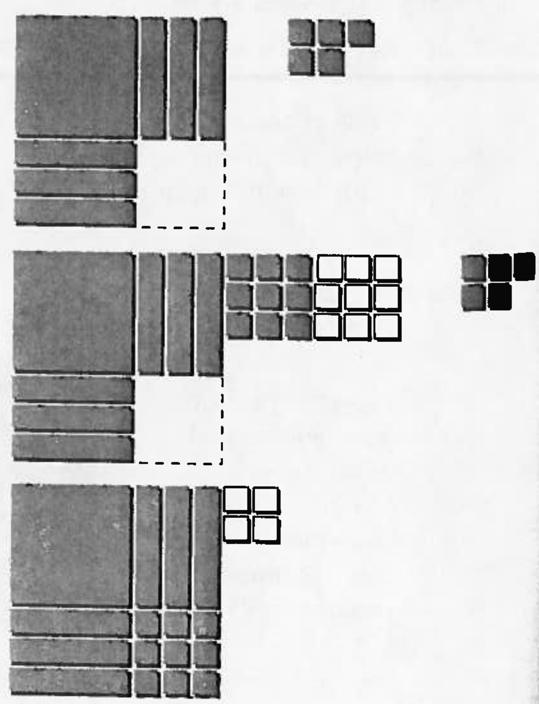
2. Repeat step 1 using the quadratic expression $x^2 + 4x + 4$.

3. Consider the quadratic expression $x^2 + 6x + 5$.

a) Describe how algebra tiles have been used to create a perfect square using the *first two terms.*

b) Explain how the relation $y = x^2 + 6x + 5$ relates to $y = (x + 3)^2 - 4$.

c) Use a graphing calculator to compare the graphs of $y = x^2 + 6x + 5$ and $y = (x + 3)^2 - 4$.



4. Consider the quadratic expression $x^2 + 4x + 3$.
   a) As in step 3, use algebra tiles or a diagram to create a perfect square using the first two terms.
   b) Explain how the relation $y = x^2 + 4x + 3$ relates to $y = (x + 2)^2 - 1$.
   c) Use a graphing calculator to compare the graphs of $y = x^2 + 4x + 3$ and $y = (x + 2)^2 - 1$.

5. Reflect Illustrate and explain how to use algebra tiles to rewrite the quadratic relation $y = x^2 + 2x + 7$ in the form $y = (x - h)^2 + k$.

The process of **completing the square** involves changing the first two terms of a quadratic relation of the form $y = ax^2 + bx + c$ into a perfect square while maintaining the balance of the original relation.

completing the square

- a process for expressing $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$

Example 1 Complete the Square

a) Rewrite $y = x^2 + 8x + 5$ in the form $y = a(x - h)^2 + k$.

b) Write the coordinates of the vertex of the parabola.

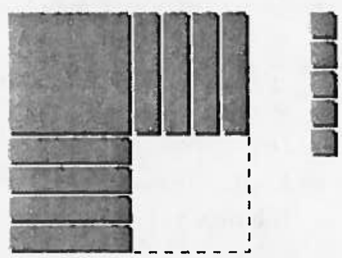
c) Sketch a graph of the relation. Label the vertex, the axis of symmetry, and two other points.

Solution

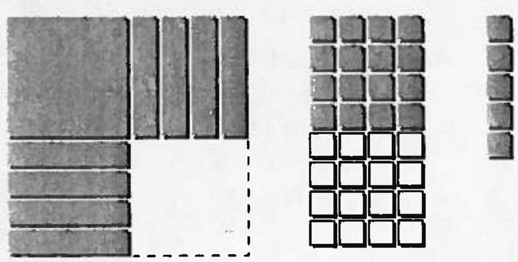
a) Method 1: Use Algebra Tiles

Create a perfect square using the first two terms in the quadratic expression $x^2 + 8x + 5$.

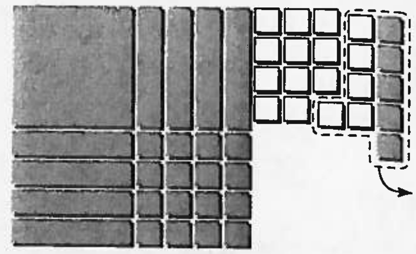
Arrange one x^2-tile and eight x-tiles so that the side lengths are equal. Place the five unit tiles to the side.



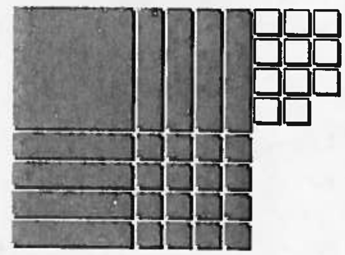
To complete the perfect square, you need to add 16 unit tiles. In order to preserve the original quadratic expression, you must also add 16 negative unit tiles.



Complete the square and collect the unit tiles. Remove zero pairs.



$x^2 + 8x + 5 = (x + 4)^2 - 11$



The relation $y = x^2 + 8x + 5$ in the form $y = a(x - h)^2 + k$ is $y = (x + 4)^2 - 11$.

Method 2: Use Algebraic Symbols

Rewrite the relation in the form $y = a(x - h)^2 + k$ by completing the square.

$y = x^2 + 8x + 5$

$= (x^2 + 8x) + 5$ — Group the first two terms.

$= (x^2 + 8x + 4^2 - 4^2) + 5$ — To make a perfect square trinomial inside the brackets, add the square of half of 8, or 4^2. To balance the equation, also subtract 4^2.

$= (x^2 + 8x + 4^2) - 4^2 + 5$ — Take -4^2 outside the brackets.

$= (x + 4)^2 - 16 + 5$ — Factor the perfect square trinomial and simplify.

$= (x + 4)^2 - 11$

The relation $y = x^2 + 8x + 5$ in the form $y = a(x - h)^2 + k$ is $y = (x + 4)^2 - 11$.

b) The vertex is (h, k), or $(-4, -11)$.

c) The equation of the axis of symmetry is $x = -4$.

To find another point on the graph, let x take any value.

Let $x = 0$.

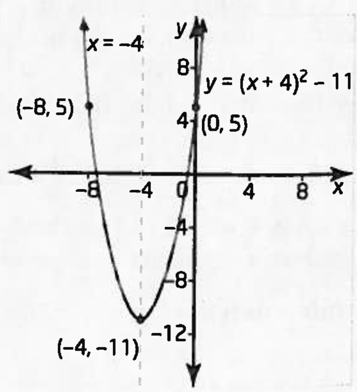
$y = 0^2 + 8(0) + 5$

$= 5$

Therefore, (0, 5) is a point on the parabola.

Due to symmetry, another point is the *partner* to this, $(-8, 5)$.

Plot the three points and complete the sketch.



Looking at the x-coordinate of this point, I know that it is located 4 units to the right of the axis of symmetry. So, its partner is located 4 units to the left of the axis of symmetry and has the same y-coordinate.

Example 2 Find a Maximum or a Minimum

Find the maximum or minimum point of the parabola with equation $y = 2x^2 + 12x + 11$.

Solution

Method 1: Complete the Square

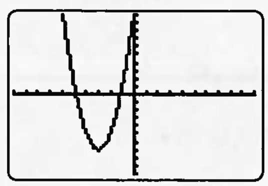
When the coefficient of the x^2-term is not 1, the first step is to factor the coefficient of x^2 from the first two terms. Then, complete the square within the brackets.

$y = 2x^2 + 12x + 11$	
$= 2(x^2 + 6x) + 11$	Factor 2 from the first two terms to make the coefficient of the x^2-term 1.
$= 2(x^2 + 6x + 3^2 - 3^2) + 11$	To make a perfect square trinomial inside the brackets, add the square of half of 6, or 3^2. Subtract the same value to balance the equation.
$= 2(x^2 + 6x + 3^2) - 2(3^2) + 11$	Take -3^2 outside of the brackets by multiplying by 2.
$= 2(x + 3)^2 - 18 + 11$	Factor the perfect square trinomial and simplify.
$= 2(x + 3)^2 - 7$	

The equation $y = 2(x + 3)^2 - 7$ is of the form $y = a(x - h)^2 + k$. The vertex is $(-3, -7)$. It is a minimum point, since a is positive.

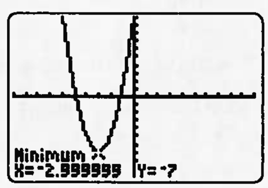
Method 2: Use a Graphing Calculator

Enter the equation using [Y=].
Press [ZOOM] and select **6:ZStandard**.



You can see that the parabola has a minimum. This is because the coefficient of x^2 is positive.

Use the Minimum operation of a graphing calculator to find the coordinates of the vertex.



- Press [2nd] [CALC] to display the **CALCULATE** menu, and select **3:minimum**.
- Move the cursor to the left of the vertex and press [ENTER].
- Move the cursor to the right of the vertex and press [ENTER].
- Move the cursor close to the vertex and press [ENTER].

The calculator will give you the approximate ordered pair that best represents the minimum point of the graph.

The minimum point is $(-3, -7)$.

Technology Tip

The TI-83 Plus or TI-84 Plus graphing calculator displays cursor coordinates as eight-character numbers, which may include a negative sign. When **Float** is selected in the **MODE** settings, X and Y are displayed with a maximum accuracy of eight digits. Since the Minimum and Maximum operations are only calculated to an accuracy of 1×10^{-5} or 0.000 01, the result displayed on the graphing calculator screen may not be accurate to all eight displayed digits.

Example 3 Path of a Ball

Reasoning and Proving
Representing
Selecting Tools
Problem Solving
Connecting
Reflecting
Communicating

The path of a ball is modelled by the equation $y = -x^2 + 2x + 3$, where x is the horizontal distance, in metres, from a fence and y is the height, in metres, above the ground.

a) What is the maximum height of the ball, and at what horizontal distance does it occur?

b) Sketch a graph to represent the path of the ball.

Solution

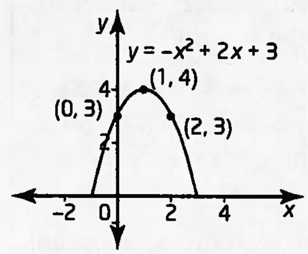
a)
$$y = -x^2 + 2x + 3$$
$$= -1(x^2 - 2x) + 3$$
Factor −1 from the first two terms.
$$= -1(x^2 - 2x + (-1)^2 - (-1)^2) + 3$$
To complete the square inside the brackets, add the square of half of −2, or $(-1)^2$. Subtract the same value to balance the equation.
$$= -1(x^2 - 2x + (-1)^2) - (-1)(-1)^2 + 3$$
Take $-(-1)^2$ outside the brackets by multiplying by −1.
$$= -1(x - 1)^2 + 1 + 3$$
Factor the perfect square trinomial and simplify.
$$= -(x - 1)^2 + 4$$

The equation $y = -(x - 1)^2 + 4$ is of the form $y = a(x - h)^2 + k$. The vertex is (1, 4). It is a maximum point since a is negative.

The maximum height of the ball is 4 m after it has been thrown a horizontal distance of 1 m.

b) The vertex is (1, 4).
When $x = 0$, $y = 3$.
By symmetry, the partner point to (0, 3) is (2, 3).

I can use these three points to graph the path of the ball. The parabola will not go below the x-axis because the height of the ball is always positive.



Example 4 Maximize Revenue

Alex runs a snowboard rental business that charges \$12 per snowboard and averages 36 rentals per day. She discovers that for each \$0.50 decrease in price, her business rents out two additional snowboards per day. At what price can Alex maximize her revenue?

Solution

Let R represent the total revenue, in dollars.
Let x represent the number of \$0.50 decreases in price.

Then, the price, in dollars, can be calculated as $12 - 0.5x$ and the number of rentals can be calculated as $36 + 2x$.

Revenue is the product of the price and the number rented.
$R = (12 - 0.5x)(36 + 2x)$

To find the maximum revenue, expand the quadratic relation, and then complete the square.

$$
\begin{aligned}
R &= (12 - 0.5x)(36 + 2x) \\
&= 432 + 6x - x^2 \\
&= -x^2 + 6x + 432 \\
&= -1(x^2 - 6x) + 432 \\
&= -1(x^2 - 6x + (-3)^2 - (-3)^2) + 432 \\
&= -1(x^2 - 6x + (-3)^2) - (-1)(-3)^2 + 432 \\
&= -1(x - 3)^2 + 9 + 432 \\
&= -(x - 3)^2 + 441
\end{aligned}
$$

I can use technology to graph both forms of the relation, $R = (12 - 0.5x)(36 + 2x)$ and $R = -(x - 3)^2 + 441$, and verify that they are equivalent.

The relation reaches a maximum value of 441 when $x = 3$.

There should be three price reductions of \$0.50 to maximize the revenue.
$12 - 0.5(3) = 10.50$

A price of \$10.50 maximizes Alex's revenue.

Key Concepts

- You can rewrite a quadratic relation of the form $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$ by completing the square.
- For a quadratic relation in the form $y = a(x - h)^2 + k$, the vertex, (h, k), represents the maximum or minimum point of the parabola. The vertex is a minimum point when $a > 0$ and a maximum point when $a < 0$.
- Completing the square allows you to find the maximum or minimum point of a quadratic relation of the form $y = ax^2 + bx + c$ algebraically.
- You can use a graphing calculator to find the maximum or minimum point by using the Maximum or Minimum operation on the graph of the quadratic relation.